Effects of Various Implicit Operators on a Flux Vector Splitting Method

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Abstract

THREE different implicit operators in a numerical method for solving the two-dimensional steady Euler equations using flux vector splitting are investigated. These include the implementation of the true Jacobian matrices in the implicit part, the use of their simplified approximate form, and a modification of the implicit part of the scheme that uses the scalar form of the implicit part based on the spectral radii of the split Jacobian matrices. The three versions of the basic algorithm are applied to several two-dimensional test cases. Special attention is paid to the stability and convergence properties.

Contents

Because the solution of inviscid, compressible flows in complicated geometries is of significant practical importance, it has been the topic of many current research projects. Although real flows in typical internal geometries are three-dimensional in nature, the basic algorithms are first conveniently developed and tested for two-dimensional cases. Recently, a major effort has been made to develop efficient methods for solving the Euler equations. Jameson et al.[1] introduced an explicit finite-volume method for solving the Euler equations for complicated two- and three-dimensional transonic flow geometries. This method uses four stage Runge-Kutta time stepping and, despite being explicit, achieves high computational efficiency because it is easy to vectorize and has been optimized for multigrid strategies.

Several methods have been introduced where flux vector splitting is implemented based on the partition of a flux vector into groups with certain specified properties. The Steger and Warming[2] version of the flux vector splitting uses plus/minus splitting in an implicit, approximate factorization scheme. Anderson et al.[3] have investigated different types of splittings applied to an implicit finite-volume method for solving the Euler equations. In some of these methods, either an explicit procedure is used or the implicit operator is simplified to increase computational efficiency. Here, the rather complicated flux Jacobian matrices to be computed at each point are replaced by their simpler approximation. However, this modification may harm the convergence properties as shown by, for example, Steger and Warming,[2] resulting in a maximum stable CFL number of $\mathcal{O}(1)$. Subsequently, Jespersen and Pulliam[4] performed stability analysis of various flux splittings of the Euler equations applied to several quasi-one-dimensional problems. They fully confirmed the experimental findings of Steger and Warming by showing that the use of the "incorrect" Jacobian matrix resulted in restrictive stability limits corresponding to CFL numbers between less than 1.0 for low Mach number subsonic cases and approximately 9.0 for supersonic cases.

The purpose of the present work is to shed more light on the problem of the correct choice of an implicit operator applied to flux vector splitting schemes in two dimensions. The explicit part of the numerical method tested in the present study is based on the finite-volume, flux vector splitting algorithm formulated for the isenthalpic Euler equations.[5]

Besides the "true" (correct) flux Jacobian matrices, two simplified implicit operators were investigated. The results obtained from the above methods were compared with calculations done with the implicit, approximate factorization, central difference scheme introduced by Beam and Warming.[6] The test cases selected for the present study included a supersonic nozzle, a supersonic diffusor, and a channel with circular arc bump with supersonic and transonic flows.

All the details of the algorithm development, including the splitting of the flux vectors F and G into, for example, F^2 and F^3, and the various implicit formulations are given in Ref. 5 and in the full version of this paper.[7] This synoptic will concentrate only on the comparison of the stability properties of the true Jacobian (TJ) and spectral radius (SR) schemes. Both are implicit Euler single-step temporal schemes; in the case of the TJ scheme, correct flux Jacobians such as $A^2 = \partial F^2/\partial Q$ are implemented. The SR scheme is a simplification of the TJ scheme obtained by replacing the true Jacobians A^2, A^3, B^2 and B^3 with their corresponding spectral radii, which results in a very simple implicit operator.

The present Fourier stability analysis is an extension of the one-dimensional approach developed by Jespersen and Pulliam.[4] Therefore, only its highlights will be presented here; for a more detailed treatment, see Ref. 4. The numerical method can be written in general operator notation as

$$M(Q^n)\Delta Q^n = P(Q^n) \tag{1}$$

Considering only perturbation about a given solution vector $\bar{Q}$, the stability of a fixed-point iteration,

$$Q^{n+1} = Q^n + M(Q^n)^{-1}P(Q^n) \tag{2}$$

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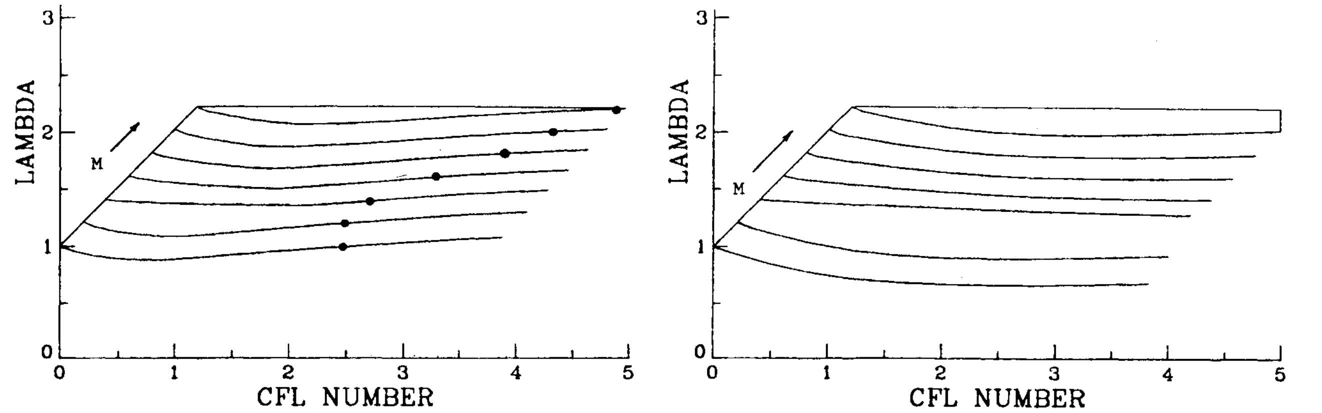

a) True Jacobian scheme.

b) Spectral radius scheme.

Fig. 1 Stability plot for the SR and TJ schemes using approximate factorization and first-order accuracy in the implicit operator and second-order accuracy in the explicit operator. The maximum eigenvalue λ_{max} is shown as a function of the two-dimensional CFL number and Mach number.

is determined by the eigenvalues of the generalized eigenvalue problem

$$L\mu = \lambda K \mu \tag{3}$$

where

$$L = M(\tilde{Q}) + \frac{\partial P}{\partial Q}(\tilde{Q}) \quad \text{and} \quad K = M(\tilde{Q})$$

For the Fourier analysis, a Fourier mode for μ,

$$\mu_{ij} = C_{\alpha,\beta} e^{I(\alpha i \Delta\xi + \beta j \Delta\eta)}, \qquad I = \sqrt{-1}$$

is assumed. Assuming constant space coefficients and simplifying $\alpha i \Delta\xi = i\theta_\xi$ and $\beta j \Delta\eta = j\theta_\eta$ leads to

$$\mu_{ij} = C e^{I(i\theta_\xi + j\theta_\eta)} \tag{4}$$

Equation (4) can now be substituted in Eq. (3), yielding the following 3×3 (in the present case) generalized eigenvalue problem:

$$L^\theta C = \lambda K^\theta C \tag{5}$$

where L^θ and K^θ are obtained from L and K by replacing the difference in L and K with difference operators acting on $e^{I(i\theta_\xi + j\theta_\eta)}$ and dividing out the same common factor. The eigenvalue equation (5) can be solved on computers by first discretizing θ_ξ and θ_η. Then, for any combination of θ_ξ and θ_η, three complex eigenvalues λ_K can be obtained from Eq. (5) using standard software built into any reasonable math library. Finally, for given CFL and Mach numbers, the amplification factor λ_{max} is obtained as a maximum norm of all λ_K for all combinations of θ_ξ and θ_η.

The stability behaviors of the TJ and SR methods were compared using a transonic flow case. Here, the flow in a channel with 10% circular arc bump was computed with inflow Mach number $M_{in} = 0.675$. The flow accelerates along the bump to supersonic speeds, followed by a shock. The results obtained from both schemes agree well with the results in Ref. 8. In this case, the SR scheme had a maximum CFL number of only 0.8. This should be expected, since the spectral radii in both directions ξ and η are only approximated. The TJ scheme was unconditionally stable, with optimum CFL number for best convergence rate between 4 and 7. Despite its more complicated form, it was clearly more efficient than the SR scheme.

The stability plots in Fig. 1 basically confirm these findings. Here, the Fourier stability analysis was applied to one point corresponding to $\xi_x = 32$, $\xi_y = 6.6$, $\eta_x = -7.3$, and $\eta_y = 38.2$; the local Mach number was increased from 0.7 to 2.4 in steps of 0.3. For each of them, the maximum norm of the complex eigenvalue $|\lambda|$ of the amplification matrix was found for a range of the two-dimensional CFL numbers between 0 and approximately 4. The range of the CFL numbers was assumed to be relatively small because the maximum stable CFL number for the SR method was between 0.8 and 2.4.

The point where the maximum eigenvalue reaches 1.0 is marked with full circle. The SR method, shown in Fig. 1a, reaches its stability limit for low Mach numbers at approximately CFL = 2.0, which corresponds to the CFL number in the ξ direction of about 1.2. With increasing Mach number, the stability limit also increases to CFL = 2.8 in the ξ direction. The overall levels of $|\lambda|$ are disturbingly high, indicating poor convergence properties. The stability analysis of the TJ scheme is shown in Fig. 1b and predicts optimum CFL numbers between 3 and 10, which are in good agreement with the present numerical experiments[7] as well as the results in Ref. 4.

In conclusion, considering the approximate nature of the Fourier stability analysis, the resulting stability limits were in surprisingly good agreement, with the limits determined by numerical experiments. The TJ scheme, as compared with the other algorithms in Ref. 7, represented a formulation that was the most consistent with the physics of the problem and consequently had the best convergence properties.

References

[1]Jameson, A., Schmidt, W., and Turkel, E., "Numerical Solutions of the Euler Equations by Finite Volume Methods Using Runge Kutta Time Stepping Schemes," AIAA Paper 81-1259, 1981.

[2]Steger, J. L. and Warming, R. F., "Flux Vector Splitting of the Inviscid Gasdynamics Equations with Application to Finite Difference Methods," *Journal of Computational Physics*, Vol. 40, 1981, pp. 263-293.

[3]Anderson, W. K., Thomas, J. L., and van Leer, B., "A Comparison of Finite Volume Flux Vector Splittings for the Euler Equations," AIAA Paper 85-0122, 1985.

[4]Jespersen, D. C. and Pulliam, T. H., "Flux Vector Splitting and Approximation Newton Methods," AIAA Paper 83-1899, 1983.

[5]von Lavante, E. and Haertl, A., "Numerical Solutions of Euler Equations Using Simplified Flux Vector Splitting," AIAA Paper 85-1333, 1985.

[6]Beam, R. M. and Warming, R. F., "An Implicit Factored Scheme for the Compressible Navier-Stokes Equations," *AIAA Journal*, Vol. 16, April 1978, pp. 393-402.

[7]von Lavante, E., Claes, D., and Anderson, W. K., "The Effects of Various Operators in a Flux Vector Splitting Method," AIAA Paper 86-0273, 1986.

[8]Ni, R. H., "A Multiple-Grid Scheme for Solving the Euler Equations," *AIAA Journal*, Vol. 20, Nov. 1982, pp. 1565-1571.